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Displaced Fock representations of the canonical commutation relations

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Abstract. We define displaced Fock representations of the canonical commutation relations in an algebraic framework. Then we consider the problem of unitary implementability of the symmetry group which acts on the one-particle space of the theory. This action induces an automorphism of the algebra of the CCR and a condition is found to ensure that this automorphism be implemented by a unitary group representation in the space of the displaced Fock representation. Using this condition, we prove the existence of infinitely many displaced Fock representations in which the group automorphism is implemented by a unitary action. This is done for a specific choice of a subgroup of the Poincaré group, and for all cases of integer value of spin.

1. Introduction

Displaced Fock states have been studied in connection with the infrared problem (Roepstorff 1970) and in connection with the quantum theory of the massless field in two-dimensional space–time (Streater and Wilde 1970). These considerations are witness to the relevance of these 'new' types of representations of the canonical commutation relations (CCR) to quantum field theory. We formulate the problem in terms of the CCR algebra. For more details on this we refer the reader to the book by Emch (1972).

For a complex vector space with an inner product $\mathcal{M}$ we can define an abstract structure $\mathfrak{A}$ called the CCR algebra over $\mathcal{M}$. With each element f of $\mathcal{M}$ we associate the (abstract) symbol $W(f)$. Next we define the product of two such objects by the formula

$$W(f)\cdot W(g)=\exp(\tfrac{1}{2}\mathrm{i}\,\mathrm{Im}(f;g))\cdot W(f+g)$$

where f, g are in $\mathcal{M}$ and $(f; g)$ denotes the inner product of f and g in $\mathcal{M}$. This relation is known as the Weyl form of the CCR. We define an adjoint operation on the symbols $W(f)$ by $W(f)^*=W(-f)$. It follows from the multiplication formula that the symbols $W(f)$ are unitary i.e. the adjoint $W(f)^*$ of $W(f)$ is the inverse of $W(f)$. Then we define finite formal sums of the $W(f)$ e.g. $c_1W(f_1)+c_2W(f_2)+\ldots+c_nW(f_n)$ where c_j are complex numbers and f_j are elements of $\mathcal{M}$. The collection of these finite formal sums is denoted by $\mathfrak{A}_0$. On $\mathfrak{A}_0$ we can define a norm $\|\cdot\|$ such that $\|A\|<\infty$ for $A\in\mathfrak{A}_0$. By its very definition, $\mathfrak{A}_0$ is a complex vector space and we complete it in the norm mentioned above, to obtain a Banach space. The norm also obeys the condition $\|AA^*\|=\|A\|^2$ called the C^* condition. The CCR algebra $\mathfrak{A}$ over $\mathcal{M}$ is then the completion of $\mathfrak{A}_0$ in the given norm. Segal (1959) proves that, up to an isomorphism, the algebra $\mathfrak{A}$ is the only

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one for which the CCR multiplication holds and for which the norm satisfies the C^* condition.

A representation of the CCR algebra is a linear function $\pi: \mathfrak{A} \rightarrow \mathfrak{B}[\mathcal{H}]$ from the algebra $\mathfrak{A}$ to the bounded operators $\mathfrak{B}(\mathcal{H})$ of some Hilbert space $\mathcal{H}$. In the representation of the CCR of the free quantum field, we represent the field φ as an unbounded operator on the Fock space. Fock space has a unit vector Ω which is called the vacuum vector, and by applying all polynomials in the field φ and its adjoint φ^*, it is possible to generate the whole of the Fock space. In the algebraic approach we are also able to construct a Hilbert space on which to represent the algebra $\mathfrak{A}$ such that there exists a vacuum vector in the space. Given that $\mathfrak{A}$ is the CCR algebra over $\mathcal{M}$, a characteristic function $E(f)$ on $\mathcal{M}$ defines a representation π of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$, such that there exists a vector Ω in $\mathcal{H}$ which is cyclic with respect to $\mathfrak{A}$ i.e. the set $\{\pi(A)\Omega: A \in \mathfrak{A}\}$ is dense in $\mathcal{H}$. Namely, any vector of $\mathcal{H}$ can be approximated to any degree of accuracy in the norm of $\mathcal{H}$ by an element of the set. The Fock representation π_0 of $\mathfrak{A}$ is given by the characteristic function $E(f) = \exp(-\frac{1}{4}\|f\|^2)$. Writing W_0 for the Fock version of the W, we note that we have the identity $(\Omega, W_0(f)\Omega) = \exp(-\frac{1}{4}\|f\|^2)$ for all $f \in \mathcal{M}$. For details on this we refer the reader to theorem 7 p 239 of Emch (1972) and to the discussion on pp 241–2 of this book.

The Hilbert space completion of $\mathcal{M}$ in the norm induced by the inner product $(\,;\,)$ is denoted by $\mathcal{K}$. $\mathcal{M}^\times$ denotes the algebraic dual of $\mathcal{M}$ i.e. the space of all linear functionals defined on $\mathcal{M}$. We denote the duality relationship by $(F; f)$ where $F \in \mathcal{M}^\times$ and $f \in \mathcal{M}$. It follows from the celebrated theorem of F Riesz (Akhiezer and Glazman 1966), that the space of continuous linear functionals on $\mathcal{M}$ can be identified with the Hilbert space completion, $\mathcal{K}$, of $\mathcal{M}$. If $F \in \mathcal{M}^\times$ we define a new representation of the CCR algebra $\mathfrak{A}$ by the formula $W_F(f) = \exp(\mathrm{i}\,\mathrm{Im}(F; f)) \cdot W_0(f)$ where $f \in \mathcal{M}$. This is called a displaced Fock representation of the CCR. This then defines the framework in which we work. In the usual framework of quantum field theory the field φ corresponds to the Fock representation and the displaced Fock representation corresponds to $\varphi + \eta$ where η is a c-number solution to the equation which φ satisfies. We now proceed to the introduction of the symmetry group.

2. The problem of implementability

In physics the space $\mathcal{M}$ mentioned in § 1 is called the one-particle space i.e. the space of wavefunctions. On this space a connected Lie group $\mathbf{G}$, called the one-particle symmetry group, acts through a unitary group representation $U(g)$. This unitary action induces an automorphism of the algebra $\sigma(g)$ through the formula

$$\sigma(g)[W(f)] = W(U(g)f)$$

where $g \in \mathbf{G}$. It can be shown that in the Fock representation of $\mathfrak{A}$ there exists a unitary group representation $V(g)$ of the group $\mathbf{G}$ on the Hilbert space $\mathcal{H}$ of the Fock representation which satisfies the relation

$$\sigma(g)[W_0(f)] = V(g)W_0(f)V(g)^{-1}.$$

This unitary representation is said to implement the action of the automorphism $\sigma(g)$. The following question now arises: in a given displaced Fock representation W_F, what is the condition on the linear functional $F \in \mathcal{M}$ which ensures that the automorphism $\sigma(g)[W_F(f)] = W_F(U(g)f)$ is implemented by the unitary group representation, $V_F(g)$, say? In order to answer this question, we begin by writing Manuceau's lemma (see lemma 1, Roepstorff 1970):

Manuceau's lemma. The two displaced Fock representations W_F and W_S, where F and S are in the algebraic dual $\mathcal{M}^\times$ of $\mathcal{M}$, are unitarily equivalent if and only if $F-S\in\mathcal{K}$ where $\mathcal{K}$ is the Hilbert space completion of $\mathcal{M}$. That is, there exists a unitary operator T on the space $\mathcal{H}$ such that $W_F(f)=T\cdot W_S(f)\cdot T^{-1}$ if and only if the condition on F and S is satisfied.

From the unitary representation U on the one-particle space $\mathcal{M}$ we can obtain a representation $U^\times$ of $\mathbf{G}$ on the space $\mathcal{M}^\times$ by exploiting the duality between $\mathcal{M}^\times$ and $\mathcal{M}$ as follows:

$$(U^\times(g)F;f)=(F;U(g)f) \qquad \text{for all } g\in\mathbf{G},$$

for all F in $\mathcal{M}^\times$ and for all f in $\mathcal{M}$. Since any element x of $\mathcal{K}$ also defines a (continuous) linear functional on $\mathcal{M}$ (indeed we have the nested sequence $\mathcal{M}\subset\mathcal{K}\subset\mathcal{M}^\times$) the representation $U^\times$ also acts in the space $\mathcal{K}$, and it can be shown that the restriction of $U^\times$ to $\mathcal{K}$ coincides with the adjoint U^* of U. We then have the following calculation

$$\begin{aligned}\sigma(g)[W_F(f)]&\\ &=W_F(U(g)f)=\exp(\mathrm{i}\,\mathrm{Im}(F;U(g)f))W_0(U(g)f)\\ &=\exp(\mathrm{i}\,\mathrm{Im}(U^\times(g)F;f))V(g)W_0(f)V(g)^{-1}\\ &=V(g)W_{U^\times(g)F}(f)V(g)^{-1}.\end{aligned}$$

It therefore follows from this calculation that the automorphism $\sigma(g)$ is unitarily implemented in the displaced Fock representation W_F if and only if the mapping $W_F(f)\mapsto W_{U^\times(g)F}(f)$ is also implemented by a unitary group representation. We now take note of Manuceau's lemma and deduce that the mapping $W_F(f)\mapsto W_{U^\times(g)F}(f)$ is implemented by a unitary operator if and only if we have the relation

$$F-U^\times(g)F\in\mathcal{K}$$

for all $g\in\mathbf{G}$. The function $\psi(g)=F-U^\times(g)F$ from the group $\mathbf{G}$ with values in $\mathcal{K}$ is an example of a 1 cocycle of $\mathbf{G}$ with coefficients in $\mathcal{H}$. Our argument has led us to associate with each displaced Fock representation W_F of the CCR algebra $\mathfrak{A}$ over $\mathcal{M}$ a 1 cocycle of $\mathbf{G}$ with coefficients in $\mathcal{K}$, the Hilbert space completion of the one-particle space $\mathcal{M}$. From a mathematical point of view it is of interest to ask whether or not this association is a bijection i.e. if $\psi(g)$ is a 1 cocycle of the form $\psi(g)=F-U^\times(g)F\in\mathcal{K}$ where $F\in\mathcal{M}^\times$, does this give rise to a displaced Fock representation in which the group $\mathbf{G}$ can be implemented by a unitary group representation? This is certainly the case, providing that all $\mathbf{G}$-invariant linear functionals vanish on $\mathcal{M}$. We do not prove this statement, but refer the reader to the argument given in Basarab-Horwath *et al* (1979). Now we are in a position to formulate the first result.

Theorem 1. If all $\mathbf{G}$-invariant functionals in $\mathcal{M}^\times$, the algebraic dual of $\mathcal{M}$, vanish on $\mathcal{M}$, then there is a one-to-one correspondence between (unitary equivalence classes of) displaced Fock representations of the CCR algebra $\mathfrak{A}$, in which the mapping $W_F(f)\mapsto W_F(U(g))$ is implemented by a unitary group representation $V_F(g)$, and 1 cocycles of $\mathbf{G}$ with coefficients in $\mathcal{K}$, of the form $F-U^\times(g)F\in\mathcal{K}$ with $F\in\mathcal{M}^\times$.

Remark. The qualification about unitary equivalence classes is a necessary mathematical remark. This is because two representations of the CCR are considered to be 'the same' when there is a unitary operator which connects them as in Manuceau's lemma. Further, if a displaced Fock representation is equivalent to the Fock representation, we say that it is trivial.

This theorem allows us to consider any relevant symmetry group. In this paper we are concerned with that subgroup of the Poincaré group which consists of (*a*) the space–time translations, (*b*) the rotations about the three axis, and (*c*) the Lorentz boost along the three axis. Unless a remark to the contrary is made, we shall call this group **G**.

3. Cocycles for spin zero

In this section we treat the spin zero representation of our group **G** and we later modify our result to all other integer spins. We construct an example of a 1 cocycle of **G** with coefficients in the space $\mathscr{H} = L^2(\mathbb{R}^3, \mathrm{d}^3\boldsymbol{p}/|\boldsymbol{p}|)$. This cocycle is not trivial in the sense that the displaced Fock representation to which it gives rise is not equivalent to the Fock representation. To verify this, we note that the Fock representation corresponds to the linear functional 0, so that the displaced Fock representation W_F is equivalent to the Fock representation if and only if $F \in \mathscr{H}$. This is a direct consequence of Manuceau's lemma. We use this criterion to prove the non-triviality of the cocycle, and hence of the resulting displaced Fock representation.

Before we begin the construction, we note a lemma which is useful in the ensuing calculation. Its proof is given in the appendix.

Lemma 2. Suppose that V is a representation of the group of real numbers $\mathbb{R}$ in the complex linear space $\mathscr{M}$, whose completion in the inner product norm is $\mathscr{H}$. The two following conditions are equivalent.

(i) $F - V^{\times}(\lambda)F \in \mathscr{H}$ for $F \in \mathscr{M}^{\times}$ and for all $\lambda \in \mathbb{R}$.

(ii) Given any $\varepsilon > 0$ then $F - V^{\times}(\lambda)F \in \mathscr{H}$ for $F \in \mathscr{M}^{\times}$ and for all $\lambda \in \mathbb{R}$ such that $|\lambda| < \varepsilon$ where $V^{\times}$ is the dual of V, acting in $\mathscr{M}^{\times}$, the algebraic dual of $\mathscr{M}$. This lemma is useful in avoiding the differential methods of Pinczon and Simon (1975).

Our Hilbert space is $\mathscr{H} = L^2(\mathbb{R}^3, \mathrm{d}^3\boldsymbol{p}/|\boldsymbol{p}|)$ and the representation of **G** which we use is the restriction to **G** of the usual representation of the Poincaré group $\mathscr{P}^{\uparrow}_{+}$ in the space $\mathscr{H}$.

We now define a sequence of cylinders $\{J_n\}$ each cylinder having its axis along the three axis.

$$J_n = \{\boldsymbol{p} \in \mathbb{R}^3 \colon \varepsilon_{n+1} \leqslant p_3 \leqslant \varepsilon_n \text{ and } p^2 = p_1^2 + p_2^2 \leqslant \varepsilon_{n+1}^2\}.$$

Here $\{\varepsilon_n\}$ is a sequence defined by recursion as follows:

$$\varepsilon_1 = 1 \qquad \text{and} \qquad \varepsilon_{n+1} = \exp(-n^2)\varepsilon_n.$$

The function f_n is defined on the cylinder J_n by

$$f_n(\boldsymbol{p}) = 1 \qquad \text{if } \boldsymbol{p} \in J_n$$

$$f_n(\boldsymbol{p}) = 0 \qquad \text{if } \boldsymbol{p} \notin J_n.$$

The norm of each function f_n in the Hilbert space $\mathscr{H}$ is finite: $\|f_n\| < \infty$. The function $\hat{f}_n = f_n/\|f_n\|$ is a unit vector in $\mathscr{H}$. Now because the cylinders J_n only meet in a two-dimensional surface, it follows that the sequence $\{\hat{f}_n\}$ is a set of orthonormal functions in $\mathcal{H}$. We now define the orthogonal direct sum of these functions: $\hat{f} = \Sigma_{n=1}^{\infty} \hat{f}_n$.

This is a well defined function and it can be shown that $\hat{f}$ is a generalised function in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^3)$. However, the function $\hat{f}$ is not in the Hilbert space $\|\hat{f}\|^2 = \Sigma_{n=1}^{\infty} \|\hat{f}_n\|^2 = \infty$.

By abuse of notation, we use the same symbol for the representation of **G** on $\hat{f}$ as for the representation of **G** in the space $\mathcal{H}$. We shall prove that $\hat{f} - U(g)\hat{f} \in \mathcal{H}$ for any element g of **G**.

The letter R will denote a rotation about the three axis. It follows from an elementary calculation, using the duality relationship, that the action $U(g)$ on $\hat{f}$ is defined by the formula $U(g)\hat{f}(\boldsymbol{p}) = \hat{f}(g\boldsymbol{p})$. Using this, it follows that $U(R)f_n = f_n$ for each f_n, whence we have that $\hat{f} - U(R)\hat{f} \in \mathcal{H}$. This is because the cylinders are invariant under rotations about the three axis. Next we deal with the Lorentz boosts along the three axis. Because these form a one-parameter group, we write $U(\lambda)$ for the unitary representation of the matrix

$$\begin{pmatrix} \cosh\lambda & 0 & 0 & \sinh\lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\lambda & 0 & 0 & \cosh\lambda \end{pmatrix}.$$

The parameter λ ranges through the whole of the real numbers, so that $U(\lambda)$ is a representation of the real numbers. Thus we may use our lemma 2, after having chosen a suitable interval about zero. In connection with this, we remark that $U(\lambda)\hat{f} - \hat{f} \in \mathcal{H}$ is equivalent to $U(-\lambda)\hat{f} - \hat{f} \in \mathcal{H}$. Combining all these remarks, we are able to prove the cocycle property of $\hat{f}$ for the Lorentz boosts in the three direction by proving $\hat{f} - U(\lambda)\hat{f} \in \mathcal{H}$ for $\lambda > 0$, where λ is chosen suitably.

We have the following calculation:

$$\|\hat{f} - U(\lambda)\hat{f}\|^2 = 2 \sum_{n=1}^{\infty} \left(1 - \sum_{m=1}^{\infty} (\hat{f}_n, U(\lambda)\hat{f}_m)\right).$$

Thus our first task is to investigate the term $\Sigma_{m=1}^{\infty} (\hat{f}_n, U(\lambda)\hat{f}_m)$, namely, we must calculate the overlap of $\hat{f}_n$ with $U(\lambda)\hat{f}_m$. The action of $U(\lambda)$ on $\hat{f}_n$ is

$$U(\lambda)\hat{f}_n(\boldsymbol{p}) = \hat{f}_n(p_1, p_2, p_3 \cosh\lambda + |\boldsymbol{p}| \sinh\lambda)$$

where $\lambda > 0$. Writing ε'_{n+1} for $\varepsilon_{n+1}\cosh\lambda + (\varepsilon^2_{n+1} + s^2)^{1/2} \sinh\lambda$ where $0 \leq s \leq \varepsilon_{n+1}$ and writing $\varepsilon'_{n+2} = \varepsilon_{n+2}\cosh\lambda + (\varepsilon^2_{n+2} + t^2)^{1/2}\sinh\lambda$ where $0 \leq t \leq \varepsilon_{n+1}$, it is not difficult to see that if $0 \leq \lambda < 0.3$ then $\varepsilon'_{n+1} < \varepsilon_n$ and $\varepsilon'_{n+2} < \varepsilon_{n+1}$. This means that we are able to choose $\lambda > 0$ so that the cylinder J_n overlaps only with a part of itself after being shifted by the boost, and a part of the cylinder J_{n+1} which has also been shifted by the boost. From this we can deduce that

$$\sum_{m=1}^{\infty} (\hat{f}_n, U(\lambda)\hat{f}_m)$$

$$= 2\pi \int_0^{\varepsilon_{n+2}} s \,\mathrm{d}s \int_{\varepsilon_{n+1}}^{\varepsilon'_{n+1}} \mathrm{d}p_3 (p_3^2 + s^2)^{-1/2} (\|f_n\| \, \|f_{n+1}\|)^{-1}$$

$$+ 2\pi \int_0^{\varepsilon_{n+1}} s\,\mathrm{d}s \int_{\varepsilon'_{n+1}}^{\varepsilon_n} \mathrm{d}p_3 (p_3^2 + s^2)^{-1/2} \|f_n\|^{-2}$$

$$>1-2\pi\int_{\varepsilon_{n+2}}^{\varepsilon_{n+1}} s\,\mathrm{d}s\int_{\varepsilon_{n+1}}^{\varepsilon'_{n+1}} \mathrm{d}p_3(p_3^2+s^2)^{-1/2}\|f_n\|^{-2}.$$

It now follows that

$$\|\hat{f}-U(\lambda)\hat{f}\|^2\leqslant\sum_{n=1}^{\infty}2\pi\int_{\varepsilon_{n-2}}^{\varepsilon_{n+1}} s\,\mathrm{d}s\int_{\varepsilon_{n+1}}^{\varepsilon'_{n+1}} \mathrm{d}p_3(p_3^2+s^2)^{-1/2}\|f_n\|^{-2}.$$

After performing some tricky integration, it is possible to show that

$$\|f_n\|^2\geqslant\pi\varepsilon_{n+1}^2(n^2-0.5)$$

and it is also possible to show that

$$\int_{\varepsilon_{n+2}}^{\varepsilon_{n+1}} s\,\mathrm{d}s\int_{\varepsilon_{n+1}}^{\varepsilon'_{n+1}} \mathrm{d}p_3(p_3^2+s^2)^{-1/2}\leqslant\varepsilon_{n+1}^2h(\lambda)$$

where $h(\lambda)<\infty$ for all $\lambda\in\mathbb{R}$. Hence, combining all this work, we have that

$$\|\hat{f}-U(\lambda)\hat{f}\|^2\leqslant\sum_{n=1}^{\infty}\frac{2h(\lambda)}{n^2-0.5}<\infty.$$

Therefore $\hat{f}$ satisfies $\hat{f}-U(\lambda)\hat{f}\in\mathscr{H}$ for the Lorentz boosts along the three directions.

The space–time translations are represented by $\exp(-\mathrm{i}\not{p}a)$ where $\not{p}a=|\boldsymbol{p}|\,|\boldsymbol{a}|-\boldsymbol{p}\cdot\boldsymbol{a}$ and the vector $(|\boldsymbol{p}|, p_1, p_2, p_3)$ is the generator of space–time translations. Using the calculations

$$\|p_j^m\hat{f}\|^2=\sum_{n=1}^{\infty}\int_{J_n}p_j^{2m}\frac{\mathrm{d}^3\boldsymbol{p}}{|\boldsymbol{p}|}\|f_n\|^{-2}\leqslant\sum_{n=1}^{\infty}\int_{J_n}|\boldsymbol{p}|\,\mathrm{d}^3\boldsymbol{p}\|f_n\|^{-2}=c<\infty$$

for $j=1,2,3$, for all $m\geqslant1$ and

$$\||\boldsymbol{p}|^m\hat{f}\|^2=\sum_{n=1}^{\infty}\int_{J_n}|\boldsymbol{p}|^{2m}\frac{\mathrm{d}^3\boldsymbol{p}}{|\boldsymbol{p}|}\|f_n\|^{-2}\leqslant\sum_{n=1}^{\infty}\int_{J_n}|\boldsymbol{p}|\frac{\mathrm{d}^3\boldsymbol{p}}{|\boldsymbol{p}|}\|f_n\|^{-2}=c<\infty$$

for all $m\geqslant1$, we deduce that

$$\hat{f}-\mathrm{e}^{\mathrm{i}\not{p}a}\hat{f}\in\mathscr{H}$$

so that the translations obey the same cocycle relation. This then completes the calculations for the component subgroups of **G**, and we now have to show that $\hat{f}-U(g)\hat{f}\in\mathscr{H}$ where g is any element of **G**. We note that any element in our group is given by a translation in space–time and the product of a Lorentz boost in the three direction with a rotation about the three axis. We write $g=(a, LR)$ where R is the rotation, L is the boost and a is the space–time translation. Thus $U(g)=\mathrm{e}^{\mathrm{i}\not{p}a}U(L)U(R)$ so that we have

$$\begin{aligned}\hat{f}-U(g)\hat{f}&=\hat{f}-\mathrm{i}^{\mathrm{i}\not{p}a}U(L)U(R)\hat{f}\\&=\hat{f}-\mathrm{e}^{\mathrm{i}\not{p}a}U(L)\hat{f}\\&=\hat{f}-\mathrm{e}^{\mathrm{i}\not{p}a}\hat{f}+\mathrm{e}^{\mathrm{i}\not{p}a}[\hat{f}-U(L)\hat{f}]\end{aligned}$$

where we have used the rotation invariance of $\hat{f}$. The two terms in the last expression are both included in the Hilbert space $\mathscr{H}$, as the previous calculations have shown. Therefore, we have proved that the function $\hat{f}$ satisfies the cocycle condition $\hat{f}-U(g)\hat{f}\in\mathscr{H}$ for any element of our group **G**. A small calculation shows that any other boost, or rotation about any other axis does not give us the required condition on $\hat{f}$. So our group

G is the largest subgroup of the Poincaré group for which $\hat{f}$ satisfies the cocycle condition.

Now suppose that c is any complex number. It is easy to see that $\hat{f}-c\hat{f}\notin\mathcal{H}$ if $c\neq 1$, so that the displaced Fock state defined by $\hat{f}$ is not unitarily equivalent to that defined by $c\hat{f}$. In this way we generate a continuous infinity of displaced Fock representations of the CCR in which the group **G** is implemented by a unitary group representation. We now go to the case of other values of spin.

4. The case of non-zero spin

We adopt, for the purpose of easier calculation, the convenient formalism of Lomont and Moses (1967) and Guillot and Petit (1966). In this formalism the group is represented on the space $\mathcal{H}$, the space which is used for the case of spin equal to zero. The Lie algebra representative of the generator of rotations about the three axis for spin $=S$ is given in this formalism by

$$J_3^S=-\mathrm{i}(\boldsymbol{p}\times\nabla)_3+S$$

or, in spherical polar coordinates, by $J_3^S=-\mathrm{i}\partial/\partial\varphi+S$. Now $-\mathrm{i}(\boldsymbol{p}\times\nabla)_3$ is the third component of the vector operator $-\mathrm{i}(\boldsymbol{p}\times\nabla)$, and is the generator of the rotations about the three axis for spin $S=0$, so that $-\mathrm{i}(\boldsymbol{p}\times\nabla)_3$ annihilates $\hat{f}$ i.e. $-\mathrm{i}(\boldsymbol{p}\times\nabla)_3\hat{f}=0$.

We can write each f_n in spherical polar coordinates, in terms of the Heaviside function

$$f_n(\boldsymbol{p})=H(\varepsilon_n-|\boldsymbol{p}|\cos\theta)H(|\boldsymbol{p}|\cos\theta-\varepsilon_{n+1})H(\varepsilon_{n+1}-|\boldsymbol{p}|\sin\theta).$$

Writing F_0 for the polar form of $\hat{f}$, it follows that

$$F_S(|\boldsymbol{p}|,\varphi,\theta)=\mathrm{e}^{-\mathrm{i}S\varphi}F_0(|\boldsymbol{p}|,\varphi,\theta)$$

is annihilated by J_3^S i.e.

$$J_3^SF_S=-\mathrm{i}\frac{\partial F_S}{\partial\varphi}+S\varphi=0.$$

Now a rotation about the three axis by an amount φ is given by the operator $\exp(\mathrm{i}J_3^S\varphi)$ so that $F_S-\exp(\mathrm{i}J_3^S\varphi)F_S=0$. Since we can write the rotation about the three axis as $U^S(\varphi)=\exp(\mathrm{i}J_3^S\varphi)$ for spin $=S$, we are then able to write $F_S-U^S(\varphi)F_S=0\in\mathcal{H}$. In the case of spin $=S$ the representation of the boost in the three direction and the representation of the space–time translations remain the same as in the case of spin equal to zero. Thus, writing $U^S(g)$ for the representation of any element g in **G** when spin $=S$, we have $F_S-U^S(g)F_S\in\mathcal{H}$ for all $g\in\mathbf{G}$.

Again noting that if $c\neq 1$, then $F_S-cF_S\notin\mathcal{H}$ so that the displaced Fock representation of the CCR generated by cF_S is not unitarily equivalent to that generated by F_S. We now have the following.

Theorem 3. For any given value of integer spin, there exist infinitely many unitarily inequivalent, non-trivial displaced Fock representations of the CCR algebra $\mathfrak{A}$, in which the group **G** is implemented by a unitary group representation.

Remark 1. Roepstorff (1970) considers the case of spin $S=1$ and shows that there exist displaced Fock representations of the CCR in which the space-time translations are implemented by a unitary group representation. We have improved this result by including the rotations about the three axis and the Lorentz boosts in this direction. We have also extended this to arbitrary integer spin. It is also possible to show that the resulting representation of $\mathbf{G}$ on the algebra of the CCR satisfies the positivity of the energy condition.

Remark 2. By using the CCR relation, it can be shown that the operator $V_{F_S}(g) = V_0(g)W_0[U^S(g)F_S - F_S]$ implements the action of $\mathbf{G}$ in the displaced Fock representation W_{F_S}, where $V_0(g)$ implements the action of the group in the Fock representation.

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Appendix. Proof of lemma 2

That (i) implies (ii) is quite clear. The converse is proved as follows. Writing $\psi(\lambda) = F - V^{\times}(\lambda)F$ we obtain

$$\begin{aligned}\psi(\lambda) &= V^{\times}(\tfrac{1}{2}\lambda)[V^{\times}(-\tfrac{1}{2}\lambda)F - V^{\times}(\tfrac{1}{2}\lambda)F] \\ &= V^{*}(\tfrac{1}{2}\lambda)[\psi(\tfrac{1}{2}\lambda) - \psi(-\tfrac{1}{2}\lambda)].\end{aligned}$$

Continuing in this way, we arrive at

$$\psi(\lambda) = A(\lambda)\psi(\lambda/2^n) + B(\lambda)\psi(-\lambda/2^n) \qquad \text{for } n > 0$$

where $A(\lambda)$ and $B(\lambda)$ are bounded operators in $\mathscr{H}$. V^* is the restriction of $V^{\times}$ to $\mathscr{H}$, and this restriction coincides with the adjoint of V. Given a fixed $\varepsilon > 0$ and a $\lambda \in \mathbb{R}$ it is always possible to find an $n > 0$ for which $|\lambda|/2^n < \varepsilon$ and this then shows that if $\psi(t) \in \mathscr{H}$ for $|t| < \varepsilon$ then $\psi(t) \in \mathscr{H}$ for all $|t| < \varepsilon$. So the lemma is proved.

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